

# Package ‘nlr’

May 9, 2026

**Type** Package

**Title** Functions for Nonlinear Least Squares Solutions - Updated 2022

**Version** 2026.4.29

**Date** 2026-04-29

**Maintainer** John C Nash <profjcnash@gmail.com>

**Description** Provides tools for working with nonlinear least squares problems. For the estimation of models reliable and robust tools than nls(), where the Gauss-Newton method frequently stops with 'singular gradient' messages. This is accomplished by using, where possible, analytic derivatives to compute the matrix of derivatives and a stabilization of the solution of the estimation equations. Tools for approximate or externally supplied derivative matrices are included. Bounds and masks on parameters are handled properly.

**License** GPL-2

**Encoding** UTF-8

**Depends** R (>= 3.5)

**Imports** digest

**Suggests** minpack.lm, optimx, numDeriv, knitr, rmarkdown, markdown, Ryacas, Deriv, microbenchmark, MASS, ggplot2, nlraa

**VignetteBuilder** knitr

**RoxygenNote** 7.2.1

**NeedsCompilation** no

**Author** John C Nash [aut, cre],  
Duncan Murdoch [aut],  
Fernando Miguez [ctb],  
Arkajyoti Bhattacharjee [ctb]

**Repository** CRAN

**Date/Publication** 2026-05-01 11:30:02 UTC

## Contents

coef.nlsr	2
fitted.nlsr	3
jaback	4
jacentral	4
jafwd	5
jand	6
model2rjfun	6
nlfb	8
nlsDeriv	11
nlsr	14
nlsr.control	15
nlsr.package	16
nlsrSS	17
nlxb	18
numericDerivR	22
nvec	23
pctrl	23
pnls	24
pnslm	24
predict.nlsr	25
print.nlsr	25
prt	26
pshort	26
rawres	27
resgr	27
resid.nlsr	28
residuals.nlsr	29
resss	29
SSlogisJN	30
summary.nlsr	31
wrapnlsr	31
<b>Index</b>	<b>34</b>

---

 coef.nlsr

*coef.nlsr*


---

### Description

prepare and display result of nlsr computations

### Usage

```
## S3 method for class 'nlsr'
coef(object, ...)
```

**Arguments**

object            an object of class `nlsr`  
...                additional data needed to evaluate the modeling functions Default FALSE

**Details**

The set of possible controls to set is as follows

**Author(s)**

J C Nash 2014-7-16 nashjc\_at\_uottawa.ca

---

`fitted.nlsr`            *fitted.nlsr*

---

**Description**

prepare and display fits of `nlsr` computations

**Usage**

```
## S3 method for class 'nlsr'  
fitted(object = NULL, data = parent.frame(), ...)
```

**Arguments**

object            an object of class `nlsr`  
data               a data frame with the data for which fits are wanted.  
...                additional data needed to evaluate the modeling functions Default FALSE

**Author(s)**

J C Nash 2014-7-16 revised 2022-11-22 nashjc\_at\_uottawa.ca

---

jaback	<i>jaback</i>
--------	---------------

---

**Description**

approximate Jacobian via forward differences

**Usage**

```
jaback(pars, resfn = NULL, bdmsk = NULL, resbest = NULL, ndstep = 1e-07, ...)
```

**Arguments**

<code>pars</code>	a named numeric vector of parameters to the model
<code>resfn</code>	a function to compute a vector of residuals
<code>bdmsk</code>	Vector defining bounds and masks. Default is NULL
<code>resbest</code>	If supplied, a vector of the residuals at the parameters <code>pars</code> to save re-evaluation.
<code>ndstep</code>	A tolerance used to alter parameters to compute numerical approximations to derivatives. Default 1e-7.
<code>...</code>	Extra information needed to compute the residuals

**Author(s)**

J C Nash 2014-7-16 nashjc\_at\_uottawa.ca

---

jacentral	<i>jacentral</i>
-----------	------------------

---

**Description**

Approximate Jacobian via central differences. Note this needs two evaluations per parameter, but generally gives much better approximation of the derivatives.

**Usage**

```
jacentral(
  pars,
  resfn = NULL,
  bdmsk = NULL,
  resbest = NULL,
  ndstep = 1e-07,
  ...
)
```

**Arguments**

pars	a named numeric vector of parameters to the model
resfn	a function to compute a vector of residuals
bdmsk	Vector defining bounds and masks. Default is NULL
resbest	If supplied, a vector of the residuals at the parameters pars to save re-evaluation.
ndstep	A tolerance used to alter parameters to compute numerical approximations to derivatives. Default 1e-7.
...	Extra information needed to compute the residuals

**Author(s)**

J C Nash 2014-7-16 revised 2022-11-22 nashjc\_at\_uottawa.ca

---

jafwd

*jafwd*


---

**Description**

approximate Jacobian via forward differences

**Usage**

```
jafwd(pars, resfn = NULL, bdmsk = NULL, resbest = NULL, ndstep = 1e-07, ...)
```

**Arguments**

pars	a named numeric vector of parameters to the model
resfn	a function to compute a vector of residuals
bdmsk	Vector defining bounds and masks. Default is NULL
resbest	If supplied, a vector of the residuals at the parameters pars to save re-evaluation.
ndstep	A tolerance used to alter parameters to compute numerical approximations to derivatives. Default 1e-7.
...	Extra information needed to compute the residuals

**Author(s)**

J C Nash 2014-7-16 nashjc\_at\_uottawa.ca

---

jand	<i>jand</i>
------	-------------

---

**Description**

approximate Jacobian via numDeriv::jacobian

**Usage**

```
jand(pars, resfn = NULL, bdmsk = NULL, resbest = NULL, ndstep = 1e-07, ...)
```

**Arguments**

pars	a named numeric vector of parameters to the model
resfn	a function to compute a vector of residuals
bdmsk	Vector defining bounds and masks. Default is NULL
resbest	If supplied, a vector of the residuals at the parameters pars to save re-evaluation.
ndstep	A tolerance used to alter parameters to compute numerical approximations to derivatives. Default 1e-7.
...	Extra information needed to compute the residuals

**Author(s)**

J C Nash 2014-7-16 nashjc\_at\_uottawa.ca

---

model2rjfun	<i>model2rjfun</i>
-------------	--------------------

---

**Description**

These functions create functions to evaluate residuals or sums of squares at particular parameter locations.

**Usage**

```
model2rjfun(modelformula, pvec, data = NULL, jacobian = TRUE, testresult = TRUE, ...)
SSmod2rjfun(modelformula, pvec, data = NULL, jacobian = TRUE, testresult = TRUE, ...)
model2ssgrfun(modelformula, pvec, data = NULL, gradient = TRUE,
               testresult = TRUE, ...)
modelexpr(fun)
```

**Arguments**

<code>modelformula</code>	A formula describing a nonlinear regression model.
<code>pvec</code>	A vector of parameters.
<code>data</code>	A dataframe, list or environment holding data used in the calculation.
<code>jacobian</code>	Whether to compute the Jacobian matrix.
<code>testresult</code>	Whether to test the function by evaluating it at <code>pvec</code> .
<code>gradient</code>	Whether to compute the gradient vector.
<code>fun</code>	A function produced by one of <code>model2rjfun</code> or <code>model2ssgrfun</code> .
<code>...</code>	Dot arguments, that is, arguments that may be supplied by <code>name = value</code> to supply information needed to compute specific quantities in the model.

**Details**

If `pvec` does not have names, the parameters will have names generated in the form 'p\_<n>', e.g. `p_1`, `p_2`. Names that appear in `pvec` will be taken to be parameters of the model.

The `data` argument may be a dataframe, list or environment, or `NULL`. If it is not an environment, one will be constructed using the components of `data` with parent environment set to be the environment of `modelformula`.

`SSmod2rjfun` returns a function with header `function(prm)`, which evaluates the residuals (and if `jacobian` is `TRUE` the Jacobian matrix) of the `selfStart` model (the `rhs` is used) at `prm`. The residuals are defined to be the right hand side of `modelformula` minus the left hand side. Note that the `selfStart` model used in the model formula must be available (i.e., loaded). If this function is called from `nlsb()` then the `control` element `japprox` must be set to value `SSJac`.

**Value**

`model2rjfun` returns a function with header `function(prm)`, which evaluates the residuals (and if `jacobian` is `TRUE` the Jacobian matrix) of the model at `prm`. The residuals are defined to be the right hand side of `modelformula` minus the left hand side.

`model2ssgrfun` returns a function with header `function(prm)`, which evaluates the sum of squared residuals (and if `gradient` is `TRUE` the gradient vector) of the model at `prm`.

`modelepr` returns the expression used to calculate the vector of residuals (and possibly the Jacobian) used in the previous functions.

**Author(s)**

John Nash and Duncan Murdoch

**See Also**

[nls](#)

## Examples

```
# We do not appear to have an example for modelexpr. See nlshr-devdoc.Rmd for one.

y <- c(5.308, 7.24, 9.638, 12.866, 17.069, 23.192, 31.443, 38.558,
      50.156, 62.948, 75.995, 91.972)

tt <- seq_along(y) # for testing

mydata <- data.frame(y = y, tt = tt)
f <- y ~ b1/(1 + b2 * exp(-1 * b3 * tt))
p <- c(b1 = 1, b2 = 1, b3 = 1)
rjfn <- model2rjfn(f, p, data = mydata)
rjfn(p)
rjfnnoj <- model2rjfn(f, p, data = mydata, jacobian=FALSE)
rjfnnoj(p)

myexp <- modelexpr(rjfn)
cat("myexp:"); print(myexp)

ssgrfn <- model2ssgrfn(f, p, data = mydata)
ssgrfn(p)

ssgrfnnoj <- model2ssgrfn(f, p, data = mydata, gradient=FALSE)
ssgrfnnoj(p)
```

---

nlfb

*nlfb: nonlinear least squares modeling by functions*

---

## Description

A simplified and hopefully robust alternative to finding the nonlinear least squares minimizer that causes 'formula' to give a minimal residual sum of squares.

## Usage

```
nlfb(
  start,
  resfn,
  jacfn = NULL,
  trace = FALSE,
  lower = -Inf,
  upper = Inf,
  weights = NULL,
  data = NULL,
  ctrlcopy = FALSE,
  control = list(),
  ...
)
```

**Arguments**

start	a numeric vector with all elements present e.g., start=c(b1=200, b2=50, b3=0.3) The start vector for this nlfb, unlike nlxb, does not need to be named.
resfn	A function that evaluates the residual vector for computing the elements of the sum of squares function at the set of parameters start. Where this function is created by actions on a formula or expression in nlxb, this residual vector will be created by evaluation of the 'model - data', rather than the conventional 'data - model' approach. The sum of squares is the same.
jacfn	A function that evaluates the Jacobian of the sum of squares function, that is, the matrix of partial derivatives of the residuals with respect to each of the parameters. If NULL (default), uses an approximation. The Jacobian MUST be returned as the attribute "gradient" of this function, allowing jacfn to have the same name and be the same code block as resfn, which may permit some efficiencies of computation.
trace	TRUE for console output during execution
lower	a vector of lower bounds on the parameters. If a single number, this will be applied to all. Default -Inf.
upper	a vector of upper bounds on the parameters. If a single number, this will be applied to all parameters. Default Inf.
weights	A vector of fixed weights or a function producing one. See the Details below.
data	a data frame of variables used by resfn and jacfn to compute the required residuals and Jacobian.
ctrlcopy	If TRUE use control supplied as is. This avoids reprocessing controls.
control	a list of control parameters. See nlsr.control().
...	additional data needed to evaluate the modeling functions

**Details**

nlfb is particularly intended to allow for the resolution of very ill-conditioned or else near zero-residual problems for which the regular nls() function is ill-suited.

This variant uses a qr solution without forming the sum of squares and cross products  $t(J)$

Neither this function nor nlxb have provision for parameter scaling (as in the parscale control of optim and package optimx). This would be more tedious than difficult to introduce, but does not seem to be a priority feature to add.

The weights argument can be a vector of fixed weights, in which case the objective function that will be minimized is the sum of squares where each residual is multiplied by the square root of the corresponding weight. Default NULL implies unit weights. weights may alternatively be a function with header function(parms, resids) to compute such a vector.

**Value**

A list of the following items:

**coefficients** A named vector giving the parameter values at the supposed solution.

**ssquares** The sum of squared residuals at this set of parameters.

**resid** The weighted residual vector at the returned parameters.

**jacobian** The jacobian matrix (partial derivatives of residuals w.r.t. the parameters) at the returned parameters.

**feval** The number of residual evaluations (sum of squares computations) used.

**jeval** The number of Jacobian evaluations used.

**weights0** The weights argument as specified.

**weights** The weights vector at the final fit.

### Author(s)

J C Nash 2014-7-16 nashjc\_at\_uottawa.ca

### Examples

```
library(nlsr)
# Scaled Hobbs problem
shobbs.res <- function(x){ # scaled Hobbs weeds problem -- residual
  # This variant uses looping
  if(length(x) != 3) stop("shobbs.res -- parameter vector n!=3")
  y <- c(5.308, 7.24, 9.638, 12.866, 17.069, 23.192, 31.443,
        38.558, 50.156, 62.948, 75.995, 91.972)
  tt <- 1:12
  res <- 100.0*x[1]/(1+x[2]*10.*exp(-0.1*x[3]*tt)) - y
}
shobbs.jac <- function(x) { # scaled Hobbs weeds problem -- Jacobian
  jj <- matrix(0.0, 12, 3)
  tt <- 1:12
  yy <- exp(-0.1*x[3]*tt)
  zz <- 100.0/(1+10.*x[2]*yy)
  jj[tt,1] <- zz
  jj[tt,2] <- -0.1*x[1]*zz*zz*yy
  jj[tt,3] <- 0.01*x[1]*zz*zz*yy*x[2]*tt
  attr(jj, "gradient") <- jj
  jj
}
st <- c(b1=2, b2=1, b3=1) # a default starting vector (named!)
# Default controls, standard Nash-Marquardt algorithm
anlf0 <- nlfb(start=st, resfn=shobbs.res, jacfn=shobbs.jac,
             trace=TRUE, control=list(prtlvl=1))
anlf0

# Hartley with step reduction factor of .2
anlf0h <- nlfb(start=st, resfn=shobbs.res, jacfn=shobbs.jac,
             trace=TRUE, control=list(prtlvl=1, lamda=0, laminc=1.0,
             lamdec=1.0, phi=0, stepredn=0.2))
anlf0h

anlf1bm <- nlfb(start=st, resfn=shobbs.res, jacfn=shobbs.jac, lower=c(2,0,0),
             upper=c(2,6,3), trace=TRUE, control=list(prtlvl=1))
```

```

anlf1bm
cat("backtrack using stepredn=0.2\n")
anlf1bmbt <- nlfb(start=st, resfn=shobbs.res, jacfn=shobbs.jac, lower=c(2,0,0),
                 upper=c(2,6,3), trace=TRUE, control=list(stepredn=0.2, prtlvl=1))
anlf1bmbt
## Short output
pshort(anlf1bm)
anlf2bm <- nlfb(start=st, resfn=shobbs.res, jacfn=shobbs.jac, lower=c(2,0,0),
                 upper=c(2,6,9), trace=TRUE, control=list(prtlvl=1))
anlf2bm
cat("backtrack using stepredn=0.2\n")

anlf2bmbt <- nlfb(start=st, resfn=shobbs.res, jacfn=shobbs.jac, lower=c(2,0,0),
                 upper=c(2,6,9), trace=TRUE, control=list(stepredn=0.2, prtlvl=1))
anlf2bmbt
## Short output
pshort(anlf2bm)

```

---

nlsDeriv

*nlsDeriv Functions to take symbolic derivatives.*


---

## Description

Compute derivatives of simple expressions symbolically, allowing user-specified derivatives.

## Usage

```
nlsDeriv(expr, name, derivEnv = sysDerivs, do_substitute = FALSE, verbose = FALSE, ...)
```

```
codeDeriv(expr, namevec, hessian = FALSE, derivEnv = sysDerivs,
          do_substitute = FALSE, verbose = FALSE, ...)
```

```
fnDeriv(expr, namevec, args = all.vars(expr), env = environment(expr),
        do_substitute = FALSE, verbose = FALSE, ...)
```

## Arguments

<code>expr</code>	An expression represented in a variety of ways. See Details.
<code>name</code>	The name of the variable with respect to which the derivative will be computed.
<code>derivEnv</code>	The environment in which derivatives are stored.
<code>do_substitute</code>	If TRUE, use <code>substitute</code> to get the expression passed as <code>expr</code> , otherwise evaluate it.
<code>verbose</code>	If TRUE, then diagnostic output will be printed as derivatives and simplifications are recognized.
<code>...</code>	Additional parameters which will be passed to <code>codeDeriv</code> from <code>fnDeriv</code> , and to <code>nlsSimplify</code> from <code>nlsDeriv</code> and <code>codeDeriv</code> .

<code>namevec</code>	Character vector giving the variable names with respect to which the derivatives will be taken.
<code>hessian</code>	Logical indicator of whether the 2nd derivatives should also be computed.
<code>args</code>	Desired arguments for the function. See Details below.
<code>env</code>	The environment to be attached to the created function. If NULL, the caller's frame is used.

## Details

Functions `nlsDeriv` and `codeDeriv` are designed as replacements for the **stats** package functions `D` and `deriv` respectively, though the argument lists do not match exactly.

The `nlsDeriv` function computes a symbolic derivative of an expression or language object. Known derivatives are stored in `derivEnv`; the default `sysDerivs` contains expressions for all of the derivatives recognized by `deriv`, but in addition allows differentiation with respect to any parameter where it makes sense. It also allows the derivative of `abs` and `sign`, using an arbitrary choice of 0 at the discontinuities.

The `codeDeriv` function computes an expression for efficient calculation of the expression value together with its gradient and optionally the Hessian matrix.

The `fnDeriv` function wraps the `codeDeriv` result in a function. If the `args` are given as a character vector (the default), the arguments will have those names, with no default values. Alternatively, a custom argument list with default values can be created using `alist`; see the example below.

The `expr` argument will be converted to a language object using `dex` (but note the different default for `do_substitute`). Normally it should be a formula with no left hand side, e.g.  $\sim x^2$ , or an expression vector e.g. `expression(x, x^2, x^3)`, or a language object e.g. `quote(x^2)`. In `codeDeriv` and `fnDeriv` the expression vector must be of length 1.

The `newDeriv` function is used to define a new derivative. The `expr` argument should match the header of the function as a call to it (e.g. as in the help pages), and the `deriv` argument should be an expression giving the derivative, including calls to `D(arg)`, which will not be evaluated, but will be substituted with partial derivatives of that argument with respect to `name`. See the examples below.

If `expr` or `deriv` is missing in a call to `newDeriv()`, it will return the currently saved derivative record from `derivEnv`. If `name` is missing in a call to `nlsDeriv` with a function call, it will print a message describing the derivative formula and return NULL.

To handle functions which act differently if a parameter is missing, code the default value of that parameter to `.MissingVal`, and give a derivative that is conditional on `missing()` applied to that parameter. See the derivatives of `"-"` and `"+"` in the file `derivs.R` for an example.

## Value

If `expr` is an expression vector, `nlsDeriv` and `nlsSimplify` return expression vectors containing the response. For formulas or language objects, a language object is returned.

`codeDeriv` always returns a language object.

`fnDeriv` returns a closure (i.e. a function).

`nlsDeriv` returns the symbolic derivative of the expression.

`newDeriv` with `expr` and `deriv` specified is called for the side effect of recording the derivative in `derivEnv`. If `expr` is missing, it will return the list of names of functions for which derivatives are recorded. If `deriv` is missing, it will return its record for the specified function.

**Note**

`newDeriv(expr, deriv, ...)` will issue a warning if a different definition for the derivative exists in the derivative table.

**Author(s)**

Duncan Murdoch

**See Also**

[deriv](#)

**Examples**

```
nlsDeriv(~ sin(x+y), "x")

f <- function(x) x^2
newDeriv(f(x), 2*x*D(x))
nlsDeriv(~ f(abs(x)), "x")

nlsDeriv(~ pnorm(x, sd=2, log = TRUE), "x")
fnDeriv(~ pnorm(x, sd = sd, log = TRUE), "x")
f <- fnDeriv(~ pnorm(x, sd = sd, log = TRUE), "x", args = alist(x =, sd = 2))
f
f(1)
100*(f(1.01) - f(1)) # Should be close to the gradient

# The attached gradient attribute (from f(1.01)) is
# meaningless after the subtraction.

# Multiple point example
xvals <- c(1, 3, 4.123)
print(f(xvals))
# Getting a hessian matrix
f2 <- ~ (x-2)^3*y - y^2
mydf2 <- fnDeriv(f2, c("x","y"), hessian=TRUE)
# display the resulting function
print(mydf2)
x <- c(1, 2)
y <- c(0.5, 0.1)
evalmydf2 <- mydf2(x, y)
print(evalmydf2)
# the first index of the hessian attribute is the point at which we want the hessian
hmat1 <- as.matrix(attr(evalmydf2,"hessian")[1,,])
print(hmat1)
hmat2 <- as.matrix(attr(evalmydf2,"hessian")[2,,])
print(hmat2)
```

nlstr

*nlstr function***Description**

Provides class nls solution to a nonlinear least squares solution using the Nash Marquardt tools.

**Usage**

```
nlstr(formula = NULL, data = NULL, start = NULL, control = NULL,
      trace = FALSE, subset = NULL, lower = -Inf, upper = Inf, weights = NULL,
      ...)
```

**Arguments**

formula	The modeling formula. Looks like 'y~b1/(1+b2*exp(-b3*T))'
data	a data frame containing data for variables used in the formula that are NOT the parameters. This data may also be defined in the parent frame i.e., 'global' to this function
start	MUST be a named vector with all elements present e.g., start=c(b1=200, b2=50, b3=0.3)
control	a list of control parameters. See nlstr.control().
trace	TRUE for console output during execution (default FALSE)
subset	an optional vector specifying a subset of observations to be used in the fitting process. NOT used currently by nlxb() or nlfb() and will throw an error if present and not NULL.
lower	a vector of lower bounds on the parameters. If a single number, this will be applied to all parameters Default -Inf.
upper	a vector of upper bounds on the parameters. If a single number, this will be applied to all parameters. Default Inf.
weights	A vector of fixed weights. The objective function that will be minimized is the sum of squares where each residual is multiplied by the square root of the corresponding weight. Default NULL implies unit weights.
...	additional data needed to evaluate the modeling functions

**Value**

A solution object of type nls

---

nlsr.control	<i>nlsr.control</i>
--------------	---------------------

---

**Description**

Set and provide defaults of controls for package nlsr

**Usage**

```
nlsr.control(control)
```

**Arguments**

control	A list of controls. If missing, the defaults are provided. See below. If a named control is provided, e.g., via a call <code>ctrlrlist&lt;- nlsr.control(japprox="jand")</code> , then that value is substituted for the default of the control in the FULL list of controls that is returned. NOTE: at 2022-6-17 there is NO CHECK FOR VALIDITY The set of possible controls to set is as follows, and is returned.
---------	--

**Value**

femax	INTEGER limit on the number of evaluations of residual function Default 10000.
japprox	CHARACTER name of the Jacobian approximation to use Default NULL, since we try to use analytic gradient
jemax	INTEGER limit on the number of evaluations of the Jacobian Default 5000
lamda	REAL initial value of the Marquardt parameter Default 0.0001 Note: mis-spelling as in JNMWS, kept as historical serendipity.
lamdec	REAL multiplier used to REDUCE lambda ( $0 < \text{lamdec} < \text{laminc}$ ) Default 4, so <code>lamda &lt;- lamda * (lamdec/laminc)</code>
laminc	REAL multiplier to INCREASE lambda ( $1 < \text{laminc}$ ) Default 10
nbtlim	if <code>stepredn &gt; 0</code> , then maximum number of backtrack loops (in addition to default evaluation); Default 6
ndstep	REAL stepsize for numerical Jacobian approximation Default $1e-7$
offset	REAL A value used to test for numerical equality, i.e. a and b are taken equal if <code>(a + offset) == (b + offset)</code> Default 100.
phi	REAL Factor used to add unit Marquardt stabilization matrix in Nash modification of LM method. Default 1
prtlvl	INTEGER The higher the value, the more intermediate output is provided. Default 0
psi	REAL Factor used to add scaled Marquardt stabilization matrix in Nash modification of LM method. Default 0
rofftest	LOGICAL If TRUE, perform (safeguarded) relative offset convergence test Default TRUE

scaleOffset	a positive constant to be added to the denominator sum-of-squares in the relative offset convergence criteria. Default 0
smallstest	LOGICAL. If TRUE tests sum of squares and terminates if very small. Default TRUE
stepredn	REAL Factor used to reduce the stepsize in a Gauss-Newton algorithm (Hartley's method). 0 means NO backtrack. Default 0
watch	LOGICAL to provide a pause at the end of each iteration for user to monitor progress. Default FALSE

**Author(s)**

J C Nash 2014-7-16 revised 2022-11-22 nashjc\_at\_uottawa.ca

---

nlr.package	<i>nlr-package Tools for solving nonlinear least squares problems The package provides some tools related to using the Nash variant of Marquardt's algorithm for nonlinear least squares. Jacobians can usually be developed by automatic or symbolic derivatives.</i>
-------------	--

---

**Description**

nlr-package

Tools for solving nonlinear least squares problems

The package provides some tools related to using the Nash variant of Marquardt's algorithm for nonlinear least squares. Jacobians can usually be developed by automatic or symbolic derivatives.

**Usage**

```
nlr.package()
```

**Details**

This package includes methods for solving nonlinear least squares problems specified by a modeling expression and given a starting vector of named parameters. Note: You must provide an expression of the form `lhs ~ rhsexpression` so that the residual expression `rhsexpression - lhs` can be computed. The expression can be enclosed in quotes, and this seems to give fewer difficulties with R. Data variables must already be defined, either within the parent environment or else in the dot-arguments. Other symbolic elements in the modeling expression must be standard functions or else parameters that are named in the start vector.

The main functions in `nlr` are:

`nlfb` Nash variant of the Marquardt procedure for nonlinear least squares, with bounds constraints, using a residual and optionally Jacobian described as R functions.

`nlxb` Nash variant of the Marquardt procedure for nonlinear least squares, with bounds constraints, using an expression to describe the residual via an R modeling expression. The Jacobian is computed via symbolic differentiation.

wrapnlsr Uses nls to solve nonlinear least squares then calls nls() to create an object of type nls. nlsr is an alias for wrapnlsr

model2rjfun returns a function with header function(prm), which evaluates the residuals (and if jacobian is TRUE the Jacobian matrix) of the model at prm. The residuals are defined to be the right hand side of modelformula minus the left hand side.

model2ssgrfun returns a function with header function(prm), which evaluates the sum of squared residuals (and if gradient is TRUE the gradient vector) of the model at prm.

modelepr returns the expression used to calculate the vector of residuals (and possibly the Jacobian) used in the previous functions.

### Author(s)

John C Nash and Duncan Murdoch

### References

Nash, J. C. (1979, 1990) *Compact Numerical Methods for Computers. Linear Algebra and Function Minimisation.* Adam Hilger./Institute of Physics Publications

Nash, J. C. (2014) *Nonlinear Parameter Optimization Using R Tools.* Wiley

---

nlsrSS

*nlsrSS - solve selfStart nonlinear least squares with nlsr package*

---

### Description

This function uses the getInitial() function to estimate starting parameters for a Gauss-Newton iteration, then calls nlsr::nls() appropriately to find a solution to the required nonlinear least squares problem.

### Usage

```
nlsrSS(formula, data)
```

### Arguments

formula	a model formula incorporating a selfStart function in the right hand side
data	a data frame with named columns that allow evaluation of the formula

### Value

A solution object of class nlsr.

List of solution elements

resid	weighted residuals at the proposed solution
jacobian	Jacobian matrix at the proposed solution

feval	residual function evaluations used to reach solution from starting parameters
jeval	Jacobian function (or approximation) evaluations used
coefficients	a named vector of proposed solution parameters
ssquares	weighted sum of squared residuals (often the deviance)
lower	lower bounds on parameters
upper	upper bounds on parameters
maskidx	vector if indices of fixed (masked) parameters
weights	specified weights on observations
formula	the modeling formula
resfn	the residual function (unweighted) based on the formula

**Author(s)**

J C Nash 2022-9-14 nashjc\_at\_uottawa.ca

---

nlxb

*nlxb: nonlinear least squares modeling by formula*


---

**Description**

A simplified and hopefully robust alternative to finding the nonlinear least squares minimizer that causes 'formula' to give a minimal residual sum of squares.

**Usage**

```
nlxb(
  formula,
  data = parent.frame(),
  start,
  trace = FALSE,
  lower = NULL,
  upper = NULL,
  weights = NULL,
  control = list(),
  ...
)
```

**Arguments**

formula	The modeling formula. Looks like 'y~b1/(1+b2*exp(-b3*T))'
data	a data frame containing data for variables used in the formula that are NOT the parameters. This data may also be defined in the parent frame i.e., 'global' to this function

start	MUST be a named vector with all elements present e.g., start=c(b1=200, b2=50, b3=0.3)
trace	TRUE for console output during execution
lower	a vector of lower bounds on the parameters. If a single number, this will be applied to all parameters Default NULL.
upper	a vector of upper bounds on the parameters. If a single number, this will be applied to all parameters. Default NULL.
weights	A vector of fixed weights or a function or formula producing one. See the Details below.
control	a list of control parameters. See nlsr.control().
...	additional data needed to evaluate the modeling functions

## Details

nlxb is particularly intended to allow for the resolution of very ill-conditioned or else near zero-residual problems for which the regular nls() function is ill-suited.

This variant uses a qr solution without forming the sum of squares and cross products  $t(J)$

Neither this function nor nlfb have provision for parameter scaling (as in the parscale control of optim and package optimx). This would be more tedious than difficult to introduce, but does not seem to be a priority feature to add.

There are many controls, and some of them are important for nlxb. In particular, if the derivatives needed for developing the Jacobian are NOT in the derivatives table, then we must supply code elsewhere as specified by the control japprox. This was originally just for numerical approximations, with the character strings "jafwd", "jaback", "jacentral" and "jand" leading to the use of a forward, backward, central or package numDeriv approximation. However, it is also possible to use code embedded in the residual function created using the formula. This is particularly useful for selfStart models, and we use the character string "SSJac" to point to such Jacobian code. Note how the starting parameter vector is found using the getInitial function from the stats package as in an example below.

The weights argument can be a vector of fixed weights, in which case the objective function that will be minimized is the sum of squares where each residual is multiplied by the square root of the corresponding weight. Default NULL implies unit weights.

weights may alternatively be a function with header function(parms, resid) to compute such a vector, or a formula whose right hand side gives an expression for the weights. Variables in the expression may include the following:

**A variable named resid** The current residuals.

**A variable named fitted** The right hand side of the model formula.

**Parameters** The parameters of the model.

**Data** Values from data.

**Vars** Variables in the environment of the formula.

**Value**

	list of solution elements
resid	weighted residuals at the proposed solution
jacobian	Jacobian matrix at the proposed solution
feval	residual function evaluations used to reach solution from starting parameters
jeval	Jacobian function (or approximation) evaluations used
coefficients	a named vector of proposed solution parameters
ssquares	weighted sum of squared residuals (often the deviance)
lower	lower bounds on parameters
upper	upper bounds on parameters
maskidx	vector if indices of fixed (masked) parameters
weights0	weights specified in function call
weights	weights at the final solution
formula	the modeling formula
resfn	the residual function (unweighted) based on the formula

**Author(s)**

J C Nash 2014-7-16 nashjc\_at\_uottawa.ca

**Examples**

```
library(nlsr)
weed <- c(5.308, 7.24, 9.638, 12.866, 17.069, 23.192, 31.443,
          38.558, 50.156, 62.948, 75.995, 91.972)
tt <- 1:12
weeddf <- data.frame(tt, weed)
frm <-
wmodu <- weed ~ b1/(1+b2*exp(-b3*tt)) # Unscaled
## nls from unit start FAILS
start1<-c(b1=1, b2=1, b3=1)
hunls1 <- try(nls(wmodu, data=weeddf, start=start1, trace=FALSE))
if (! inherits(hunls1, "try-error")) print(hunls1) ## else cat("Failure -- try-error\n")
## nlxb from unit start
hunlx1 <- try(nlxb(wmodu, data=weeddf, start=c(b1=1, b2=1, b3=1), trace=FALSE))
if (! inherits(hunlx1, "try-error")) print(hunlx1)

st2h<-c(b1=185, b2=10, b3=.3)
#' hunls2 <- try(nls(wmodu, data=weeddf, start=st2h, trace=FALSE))
if (! inherits(hunls1, "try-error")) print(hunls1) ## else cat("Failure -- try-error\n")
## nlxb from unit start
hunlx1 <- try(nlxb(wmodu, data=weeddf, start=st2h, trace=FALSE))
if (! inherits(hunlx1, "try-error")) print(hunlx1)

# Functional models need to use a Jacobian approximation or external calculation.
# For example, the SSlogis() selfStart model from \code{stats} package.
```

```

# nls() needs NO starting value
hSSnls<-try(nls(weed~SSlogis(tt, Asym, xmid, scal), data=weeddf))
summary(hSSnls)
# We need to get the start for nlxb explicitly
stSS <- getInitial(weed ~ SSlogis(tt, Asym, xmid, scal), data=weeddf)
hSSnlx<-try(nlxb(weed~SSlogis(tt, Asym, xmid, scal), data=weeddf, start=stSS))
hSSnlx

# nls() can only bound parameters with algorithm="port"
# and minpack.lm is unreliable in imposing bounds, but nlsr copes fine.
lo<-c(0, 0, 0)
up<-c(190, 10, 2) # Note: start must be admissible.
bnls0<-try(nls(wmodu, data=weeddf, start=st2h,
              lower=lo, upper=up)) # should complain and fail

bnls<-try(nls(wmodu, data=weeddf, start=st2h,
              lower=lo, upper=up, algorithm="port"))
summary(bnls)
bnlx<-try(nlxb(wmodu, data=weeddf, start=st2h, lower=lo, upper=up))
bnlx

# nlxb() can also MASK (fix) parameters. The mechanism of maskidx from nls
# is NO LONGER USED. Instead we set upper and lower parameters equal for
# the masked parameters. The start value MUST be equal to this fixed value.
lo<-c(190, 0, 0) # mask first parameter
up<-c(190, 10, 2)
strt <- c(b1=190, b2=1, b3=1)
mnlx0<-try(nlxb(wmodu, start=strt, data=weeddf,
               lower=lo, upper=up))
mnlx0
mnlx<-try(nls(wmodu, data=weeddf, start=strt,
              lower=lo, upper=up, algorithm="port"))
summary(mnlx)

# Try first parameter masked and see if we get SEs
lo<-c(200, 0, 0) # mask first parameter
up<-c(100, 10, 5)
strt <- c(b1=200, b2=1, b3=1)
mnlx1<-try(nlxb(wmodu, start=strt, data=weeddf,
               lower=lo, upper=up))
mnlx1
mnlx2<-try(nls(wmodu, data=weeddf, start=strt,
               lower=lo, upper=up, algorithm="port"))
summary(mnlx2)

# Try with weights on the observations
mnlx3<-try(nlxb(wmodu, start=strt, data=weeddf,
               weights = ~ 1/weed))
mnlx3

```

---

numericDerivR      *numericDerivR: numerically evaluates the gradient of an expression.  
All in R*

---

### Description

This version is all in R to replace the C version in package stats

### Usage

```
numericDerivR(
  expr,
  theta,
  rho = parent.frame(),
  dir = 1,
  eps = .Machine$double.eps^(1/if (central) 3 else 2),
  central = FALSE
)
```

### Arguments

expr	expression or call to be differentiated. Should evaluate to a numeric vector.
theta	character vector of names of numeric variables used in expr.
rho	environment containing all the variables needed to evaluate expr.
dir	numeric vector of directions, typically with values in -1, 1 to use for the finite differences; will be recycled to the length of theta.
eps	a positive number, to be used as unit step size hh for the approximate numerical derivative $(f(x+h)-f(x))/h$ or the central version, see central.
central	logical indicating if central divided differences should be computed, i.e., $(f(x+h) - f(x-h)) / 2h$ . These are typically more accurate but need more evaluations of $f(x)$ .

### Value

The value of `eval(expr, envir = rho)` plus a matrix attribute "gradient". The columns of this matrix are the derivatives of the value with respect to the variables listed in theta.

### Examples

```
ex <- expression(a/(1+b*exp(-tt*c)) - weed)
weed <- c(5.308, 7.24, 9.638, 12.866, 17.069, 23.192, 31.443,
          38.558, 50.156, 62.948, 75.995, 91.972)
tt <- 1:12
a <- 200; b <- 50; c <- 0.3
dhobb <- numericDerivR(ex, theta=c("a", "b", "c"))
print(dhobb)
# exf <- ~ a/(1+b*exp(-tt*c)) - weed
```

```
# Note that a formula doesn't work
# dh1 <- try(numericDerivR(exf, theta=c("a", "b", "c")))
```

---

nvec	<i>nvec</i>
------	-------------

---

**Description**

Compact display of a specified named vector

**Usage**

```
nvec(vec)
```

**Arguments**

vec                    a named vector of parameters

**Value**

none (Note that we may want to change this.)

**Author(s)**

J C Nash 2014-7-16 nashjc\_at\_uottawa.ca

---

pctrl	<i>pctrl</i>
-------	--------------

---

**Description**

Compact display of specified control vector for package nlsr.

**Usage**

```
pctrl(control)
```

**Arguments**

control                a control list

**Value**

none

**Author(s)**

J C Nash 2014-7-16 nashjc\_at\_uottawa.ca

---

*pnls*

*pnls*

---

**Description**

Compact display of specified *nls* object *x*

**Usage**

`pnls(x)`

**Arguments**

*x* an *nls()* result object from *nls()* or *nlsLM()*

**Value**

none

**Author(s)**

J C Nash 2014-7-16, 2023-5-8 nashjc \_at\_ uottawa.ca

---

*pnlslm*

*pnlslm*

---

**Description**

Compact display of specified *nls.lm* object *x*. This code returns the iteration count in a different variable from that of *nls* objects.

**Usage**

`pnlslm(x)`

**Arguments**

*x* an *nls()* result object from `minpack.lm::nls.lm()`

**Value**

none

**Author(s)**

J C Nash 2014-7-16, 2023-5-8 nashjc \_at\_ uottawa.ca

---

predict.nlsr	<i>predict.nlsr</i>
--------------	---------------------

---

**Description**

prepare and display predictions from an nlsr model

**Usage**

```
## S3 method for class 'nlsr'
predict(object = NULL, newdata = list(), ...)
```

**Arguments**

object	an object of class nlsr
newdata	a dataframe containing columns that match the original dataframe used to estimate the nonlinear model in the nlsr object
...	additional data needed to evaluate the modeling functions Default FALSE

**Author(s)**

J C Nash 2014-7-16 nashjc\_at\_uottawa.ca

---

print.nlsr	<i>print.nlsr</i>
------------	-------------------

---

**Description**

prepare and display result of nlsr computations

**Usage**

```
## S3 method for class 'nlsr'
print(x, ...)
```

**Arguments**

x	an object of class nlsr
...	additional data needed to evaluate the modeling functions Default FALSE

**Details**

The set of possible controls to set is as follows

**Author(s)**

J C Nash 2014-7-16 nashjc\_at\_uottawa.ca

prt

*prt*

---

**Description**

To display the calling name of x and print the object with the print.nlsr() function.

**Usage**

prt(x)

**Arguments**

x                    an object of class nlsr

**Author(s)**

J C Nash 2022-11-22 nashjc\_at\_uottawa.ca

---

pshort

*pshort*

---

**Description**

To provide a 1-line summary of an object of class nlsr.

**Usage**

pshort(x)

**Arguments**

x                    an object of class nlsr

**Author(s)**

J C Nash 2022-11-22 nashjc\_at\_uottawa.ca

---

rawres	<i>rawres</i>
--------	---------------

---

**Description**

Prepare and display raw residuals of nlsr computations NOTE: we use model - data form i.e., rhs - lhs

**Usage**

```
rawres(object = NULL, data = parent.frame(), ...)
```

**Arguments**

object	an object of class nlsr
data	a data frame with the data for which fits are wanted
...	additional data needed to evaluate the modeling functions

**Value**

A vector of the raw residuals

**Author(s)**

J C Nash 2014-7-16 revised 2022-11-22 nashjc\_at\_uottawa.ca

---

resgr	<i>resgr</i>
-------	--------------

---

**Description**

Computes the gradient of the sum of squares function for nonlinear least squares where resfn and jacfn supply the residuals and Jacobian

**Usage**

```
resgr(prm, resfn, jacfn, ...)
```

**Arguments**

prm	a numeric vector of parameters to the model
resfn	a function to compute a vector of residuals
jacfn	a function to compute the Jacobian of the sum of squares. If the value is quoted, then the function is assumed to be a numerical approximation. Currently one of "jafwd", "jaback", "jacentral", or "jand".
...	Extra information needed to compute the residuals

**Details**

Does NOT (yet) handle calling of code built into selfStart models. That is, codes that in nlxb employ control japprox="SSJac".

**Value**

The main object returned is the numeric vector of residuals computed at prm by means of the function resfn. There are Jacobian and gradient attributes giving the Jacobian (matrix of 1st partial derivatives whose row i contains the partial derivative of the i'th residual w.r.t. each of the parameters) and the gradient of the sum of squared residuals w.r.t. each of the parameters. Moreover, the Jacobian is repeated within the gradient attribute of the Jacobian. This somewhat bizarre structure is present for compatibility with nls() and some other legacy functions, as well as to simplify the call to nlfb().

**Author(s)**

J C Nash 2014-7-16 revised 2022-11-22 nashjc \_at\_ uottawa.ca

---

resid.nlsr	<i>resid.nlsr</i>
------------	-------------------

---

**Description**

prepare and display result of nlsr computations

**Usage**

```
## S3 method for class 'nlsr'
resid(object, ...)
```

**Arguments**

object	an object of class nlsr
...	additional data needed to evaluate the modeling functions

**Author(s)**

J C Nash nashjc \_at\_ uottawa.ca

```
### remove _at_export to try to overcome NAMESPACE issue
```

---

residuals.nlsr	<i>residuals.nlsr</i>
----------------	-----------------------

---

**Description**

prepare and display result of nlsr computations

**Usage**

```
## S3 method for class 'nlsr'
residuals(object, ...)
```

**Arguments**

object	an object of class nlsr
...	additional data needed to evaluate the modeling functions

**Author(s)**

J C Nash nashjc \_at\_ uottawa.ca

---

resss	<i>resss</i>
-------	--------------

---

**Description**

compute the sum of squares from resfn at parameters prm

**Usage**

```
resss(prm, resfn, ...)
```

**Arguments**

prm	a named numeric vector of parameters to the model
resfn	a function to compute a vector of residuals
...	Extra information needed to compute the residuals

**Author(s)**

J C Nash 2014-7-16 nashjc \_at\_ uottawa.ca

---

SSlogisJN

*Alternative self start for three-parameter logistic function SSlogis*


---

### Description

Self starter for a 3-parameter logistic function.

The equation for this function is:

$$f(input) = Asym / (1 + \exp((xmid - input) / scal))$$

The approach of the function SSlogis() in base R uses a different algorithm and returns the actual solution rather than starting parameters, so runs a complete set of iterations, only to try to repeat from the solution with the standard algorithm.

### Usage

```
SSlogisJN(input, Asym, xmid, scal)
```

### Arguments

input	input vector (input)
Asym	asymptotic value for large values of x
xmid	a numeric parameter representing the x value at the inflection point of the curve. The value of SSlogisJN will be Asym/2 at xmid.
scal	numeric scale parameter on the input axis

### References

Ratkowsky, David A. (1983) Nonlinear Regression Modeling, A Unified Practical Approach, Dekker: New York, section 8.3.2

### Examples

```
{
## require(ggplot2)
require(nlsr)
set.seed(1234)
x <- seq(0, 20, .2)
y <- SSlogisJN(x, 5, 10, .5) + rnorm(length(x), 0, 0.15)
frm<-y ~ SSlogisJN(x, Asym, xmid, scal)
dat <- data.frame(x = x, y = y)
strt<-getInitial(frm, dat)
cat("Proposed start:\n"); print(strt)
fit <- nlxb(frm, strt, data = dat, control=list(japprox="SSJac"))
print(fit)
## plot
## ggplot(data = dat, aes(x = x, y = y)) +
```

```
## geom_point() +
## geom_line(aes(y = fitted(fit)))
}
```

---

summary.nlsr

*summary.nlsr*


---

### Description

prepare display result of nlsr computations - NOT compact output

### Usage

```
## S3 method for class 'nlsr'
summary(object, ...)
```

### Arguments

object            an object of class nlsr  
 ...              additional data needed to evaluate the modeling functions

### Details

The set of possible controls to set is as follows

### Author(s)

J C Nash 2014-7-16 nashjc\_at\_uottawa.ca

---

wrapnlsr

*wrapnlsr*


---

### Description

Provides class nls solution to a nonlinear least squares solution using the Nash Marquardt tools.

### Usage

```
wrapnlsr(formula = NULL, data = NULL, start = NULL, control = NULL,
          trace = FALSE, subset = NULL, lower = -Inf, upper = Inf, weights = NULL,
          ...)
```

**Arguments**

formula	The modeling formula. Looks like 'y~b1/(1+b2*exp(-b3*T))'
data	a data frame containing data for variables used in the formula that are NOT the parameters. This data may also be defined in the parent frame i.e., 'global' to this function
start	MUST be a named vector with all elements present e.g., start=c(b1=200, b2=50, b3=0.3)
control	a list of control parameters. See nlsr.control().
trace	TRUE for console output during execution (default FALSE)
subset	an optional vector specifying a subset of observations to be used in the fitting process. NOT used currently by nlxb() or nlfb() and will throw an error if present and not NULL.
lower	a vector of lower bounds on the parameters. If a single number, this will be applied to all parameters Default -Inf.
upper	a vector of upper bounds on the parameters. If a single number, this will be applied to all parameters. Default Inf.
weights	A vector of (usually fixed) weights. The objective function that will be minimized is the sum of squares where each residual is multiplied by the square root of the corresponding weight. Default NULL implies unit weights.
...	additional data needed to evaluate the modeling functions

**Value**

A solution object of type nls

**Examples**

```
library(nlsr)
cat("kvanderpoel.R test of wrapnlr\n")
x<-c(1,3,5,7)
y<-c(37.98,11.68,3.65,3.93)
pks28<-data.frame(x=x,y=y)
fit0<-try(nls(y~(a+b*exp(1)^(-c*x)), data=pks28, start=c(a=0,b=1,c=1),
            trace=TRUE))
print(fit0)
fit1<-nlxb(y~(a+b*exp(-c*x)), data=pks28, start=c(a=0,b=1,c=1), trace = TRUE)
print(fit1)
cat("\n\n or better\n")
fit2<-wrapnlr(y~(a+b*exp(-c*x)), data=pks28, start=c(a=0,b=1,c=1),
             lower=-Inf, upper=Inf, trace = TRUE)
fit2

weed <- c(5.308, 7.24, 9.638, 12.866, 17.069, 23.192, 31.443,
          38.558, 50.156, 62.948, 75.995, 91.972)
tt <- 1:12
weeddf <- data.frame(tt, weed)
hobbsu <- weed ~ b1/(1+b2*exp(-b3*tt))
```

```
st2 <- c(b1=200, b2=50, b3=0.3)
wts <- 0.5^tt # a straight scaling comes via wts <- rep(0.01, 12)
lo <- c(200, 0, 0)
up <- c(1000, 1000, 1000)
whuw2 <- try(wrapnlr(start=st2, formula=hobbsu, data=weeddf, subset=2:11,
                  weights=wts, trace=TRUE, lower=lo, upper=up))
summary(whuw2)
deviance(whuw2)
whuw2a <- try(nlsr(start=st2, formula=hobbsu, data=weeddf, subset=2:11,
                 weights=wts, trace=TRUE, lower=lo, upper=up))
summary(whuw2a)
deviance(whuw2a)
```

# Index

- \* **jacobian**
  - nlsDeriv, 11
- \* **nls**
  - nlsr.package, 16
- \* **nonlinear&least&squares**
  - nlsr.package, 16
- \* **nonlinear**
  - model2rjfun, 6
- \*
  - model2rjfun, 6
  - nlsDeriv, 11
- alist, 12
- codeDeriv (nlsDeriv), 11
- coef.nlsr, 2
- D, 12
- deriv, 12, 13
- dex, 12
- fitted.nlsr, 3
- fnDeriv (nlsDeriv), 11
- jaback, 4
- jacentral, 4
- jafwd, 5
- jand, 6
- model2rjfun, 6
- model2ssgrfun (model2rjfun), 6
- modelexpr (model2rjfun), 6
- nlf, 8
- nls, 7
- nlsDeriv, 11
- nlsr, 14
- nlsr.control, 15
- nlsr.package, 16
- nlsrSS, 17
- nlsr, 18
- numericDerivR, 22
- nvec, 23
- pctrl, 23
- pnls, 24
- pnlsml, 24
- predict.nlsr, 25
- print.nlsr, 25
- prr, 26
- pshort, 26
- rawres, 27
- resgr, 27
- resid.nlsr, 28
- residuals.nlsr, 29
- resss, 29
- SSlogisJN, 30
- SSmod2rjfun (model2rjfun), 6
- substitute, 11
- summary.nlsr, 31
- wrapnlsr, 31